THE MCKAY-THOMPSON SERIES ASSOCIATED WITH THE IRREDUCIBLE CHARACTERS OF THE MONSTER

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Abstract. Let $\mathbb{V} = \coprod_{\eqsim=\nvdash}^{\infty} \mathbb{V}_{\eqsim}$ be the graded monster module of the monster simple group $\mathbb{M}$ and let χ_k be an irreducible representation of $\mathbb{M}$. The generating function of c_{hk} (the multiplicity of χ_k in $\mathbb{V}_{\eqsim}$) is determined. Furthermore, the invariance group of the modular function associated with the generating function is also determined in this paper.

1. Introduction

Let $\mathbb{M}$ be the monster simple group and $\mathbb{V}$ be the monster module of Frenkel-Lepowsky-Meurman [3]. $\mathbb{V}$ is a graded $\mathbb{M}$ module

$$\mathbb{V} = \coprod_{\eqsim=\nvdash}^{\infty} \mathbb{V}_{\eqsim}$$

such that

$$j(q) - 744 = \sum_{h=0}^{\infty} \dim \mathbb{V}_{\eqsim} \shortparallel^{\eqsim - \nVdash}.$$

In particular, $\dim \mathbb{V}_{\nvdash} = \nVdash$, $\dim \mathbb{V}_{\nVdash} = \nvdash$, $\dim \mathbb{V}_{\nvDash} = \nVdash\nrightarrow\ntriangleleft\nleftarrow\nleftarrow\ntrianglerighteq$, $\dim \mathbb{V}_{\nVdash} = \nvDash\nVdash\ntrianglerighteq\nrightarrow\nVdash\ntriangleright\ntriangleleft\nvdash$, $\cdots$. Let χ_k, $1 \leq k \leq 194$, be the irreducible characters of $\mathbb{M}$, which will often be used to denote the irreducible representations also. For the first few $\mathbb{V}_{\beth}$'s, we have the decompositions :

$$\mathbb{V}_{\nvdash} = \chi_{\nVdash},$$
$$\mathbb{V}_{\nvDash} = \chi_{\nVdash} + \chi_{\nvDash},$$
$$\mathbb{V}_{\nVdash} = \chi_{\nVdash} + \chi_{\nvDash} + \chi_{\nVdash}.$$

In general, write

$$\mathbb{V}_{\eqsim} = \sum_{\daleth=\nVdash}^{\nVdash\nrightarrow\ntrianglerighteq} \eqsim\daleth \chi\daleth$$

where c_{hk} is the multiplicity of χ_k in $\mathbb{V}_{\eqsim}$. The table of c_{hk} for $0 \leq h \leq 51$, $1 \leq k \leq 194$ can be found in McKay-Strauss [6].

We also list some of the multiplicities c_{h1} of the trivial character χ_1 in V_h.

h	0	2	3	4	5	6	7	8	9	10	11	$\cdots$	20	$\cdots$	30	$\cdots$	40	$\cdots$	50
c_{h1}	1	1	1	2	2	4	4	7	8	12	14	$\cdots$	167	$\cdots$	1762	$\cdots$	15913	$\cdots$	129734

Let us consider, for each irreducible character χ_k, the generating function :

$$t_{\chi_k}(x) = x^{-1} \sum_{h=0}^{\infty} c_{hk} x^h.$$

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The multiplicity c_{hk} can be computed as follows :

$$c_{hk} = \frac{1}{|\mathbb{M}|}\sum_{g\in\mathbb{M}} Tr(g|\mathbb{V}_{\approx})\chi_{\neg}(\eth).$$

Therefore the generating function of c_{hk} is

$$t_{\chi_k}(x) = x^{-1}\sum_{h=1}^{\infty}\sum_{g\in\mathbb{M}}\frac{1}{|\mathbb{M}|}Tr(g|\mathbb{V}_{\approx})\chi_{\neg}(\eth)\frown^{\approx}.$$

If we replace the indeterminate x by $q = e^{2\pi iz}$, $z \in \mathbb{H} = \{z \in \mathbb{C}|\mathrm{Im}(F) > \nvdash\}$ then

$$t_{\chi_k}(q) = \frac{1}{|\mathbb{M}|}\sum_{g\in\mathbb{M}}\chi_k(g)t_g(q)$$

where

$$t_g(q) = q^{-1}\sum_{h=0}^{\infty} Tr(g|\mathbb{V}_{\approx})\shortparallel^{\approx}$$

is the McKay-Thompson series for the element g in $\mathbb{M}$. Thus $t_\chi(q)$ for the irreducible character χ is the weighted sum of the McKay-Thompson series for the element g of $\mathbb{M}$. Not all $t_\chi(q)$'s are distinct and in fact there are exactly 172 distinct $t_\chi(q)$'s, since

$$t_\chi(q) = t_{\bar\chi}(q)$$

where $\bar\chi$ is the complex conjugate of χ and there is no other equalities among $t_\chi(q)$'s. One of the obvious questions one will raise here will be :

Problem. Determine the invariance group Γ_χ of $t_\chi(q)$.

Here Γ_χ is defined to be :

Definition. $\Gamma_\chi = \{A \in SL_2(\mathbb{R})|\approx_\chi(\mathbb{A}F) = \approx_\chi(F)\}$.

Since $t_\chi(z)$ is a modular function, Γ_χ is a discrete subgroup of $SL_2(\mathbb{R})$. Let us here review some of the properties of the invariance group Γ_g of the McKay-Thompson series $t_g(z)$ for the element $g \in \mathbb{M}$.

(0). For $G \subset GL_2^+(\mathbb{R})$, $\bar G$ is the image of G in $PGL_2^+(\mathbb{R})$.

(1). $\Gamma_0(N) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})| \equiv \nvdash \pmod N\ \}$.

(2). For an exact divisor $e||N$ (i.e. $e|N$ and $\gcd(e, \frac{N}{e}) = 1$) let

$$W_e = \begin{pmatrix} ae & b \\ cN & de \end{pmatrix}, \quad a,\, b,\, c,\, d \in \mathbb{Z}, \quad \eth^{\nvDash} - \mathbb{N} = .$$

Then $\bar W_e$ normalizes $\bar\Gamma_0(N)$ and $\bar W_e^2 \in \bar\Gamma_0(N)$.

(3). Let h be a divisor of n. Then $n|h + e, f, \cdots$ is defined to be

$$\begin{pmatrix} \frac{1}{h} & 0 \\ 0 & 1 \end{pmatrix}\langle\Gamma_0(\frac{n}{h}), W_e, W_f, \cdots\rangle\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}.$$

(4). For each g in $\mathbb{M}$, Γ_g, the invariance group of $t_g(z)$, is a normal subgroup of index h_g in $n_g|h_g + e_g, f_g, \cdots$, the eigen group of g [2]. Note that for each A in $n_g|h_g + e_g, f_g, \cdots$, $(t_g|A)(z) = \sigma t_g(z)$ where σ is an h_g-th root of unity. We will often use n, h, e, f, etc. instead of n_g, h_g e_g, f_g, etc. for simplicity.

(5). Γ_g contains $\Gamma_0(n_g h_g)$.

For each irreducible character of $\mathbb{M}$, we now define :

Definition. $N_\chi = \text{lcm}\{n_g h_g | g \in \mathbb{M}, \chi(\eth) \neq \nvdash\}$.

It is obvious that $t_\chi(z)$ is invariant under $\Gamma_0(N_\chi)$. Note that N_χ can be quite large ($N_{\chi_1} = 2^6 3^3 5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$) or relatively small ($N_{\chi_{166}} = 2^6 3^2 7 = 4032$).

The purpose of this paper is to show

Theorem. $\Gamma_\chi = \Gamma_0(N_\chi)$.

2. Poles of $t_\chi(z)$

For each cusp c in $\mathbb{Q} \cup \{\infty\}$, we define Φ_c to be the set $\Phi_c = \{g \in \mathbb{M} |$ is equivalent to ∞ in $\Gamma_g\}$ and decompose $t_\chi(z)$ into :

$$\frac{1}{|\mathbb{M}|} \sum_{g \in \Phi_c} \chi(g) t_g(z) + \frac{1}{|\mathbb{M}|} \sum_{g \notin \Phi_c} \chi(g) t_g(z).$$

Since the McKay-Thompson series $t_g(z)$ is a generator of the function field of the compact Riemann surface $\Gamma_g \backslash \mathbb{H}^*$ ($\mathbb{H}^* = \mathbb{H} \cup \{\infty\}$) of genus 0 and has a unique pole at ∞ (and at all cusps $c \in \mathbb{Q}$ equivalent to ∞ in Γ_g). Obviously, $\frac{1}{|\mathbb{M}|} \sum_{g \notin \Phi_c} \chi(g) t_g(z)$ is holomorphic at c. Hence, whether c is a pole of $t_\chi(z)$ or not is determined by the singular part of

$$\frac{1}{|\mathbb{M}|} \sum_{g \in \Phi_c} \chi(g) t_g(z)$$

at c. For example, if $c = \infty$, then ∞ is a pole of $t_\chi(z)$ if and only if χ is the trivial character since the singular part of $t_\chi(z)$ at ∞ is given by $\frac{1}{|\mathbb{M}|} \sum_{g \in \mathbb{M}} \chi(g) \frac{1}{q}$ and

$$\sum_{g \in \mathbb{M}} \chi_i(g) = \begin{cases} |\mathbb{M}| & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

Suppose $c \in \mathbb{Q}$. For each g in Φ_c, let $A \in SL_2(\mathbb{Z})$ be chosen so that $A\infty = c$. Then $t_g(Az) = (t_g|A)(z)$ has an expansion in $q = e^{2\pi i z}$ of the form

$$a q^{-\frac{1}{\mu}} + \cdots$$

where $\mu = [\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle : A^{-1} \Gamma_0(n_g h_g)_c A]$, where the subscript c denotes the stabilizer. We contend that the contribution of the $t_g(z)$ to the singular part of $t_\chi(z)$ is

$$a q^{-\frac{1}{\mu}}.$$

Indeed, by our assumption the cusp c is equivalent to ∞ in Γ_g and so there is $B \in \Gamma_g$ such that $B\infty = c$ and

$$(t_g|B)(q) = t_g(q) = q^{-1} + \sum_{i \geq 0} a_i q^i.$$

The only difference between $(t_g|A)(z)$ and $(t_g|B)(z)$ lies in the power of q and a, hence the contention.

In order to determine whether c is pole of $t_\chi(z)$ or not, one has to :

(1). Determine whether c is equivalent to ∞ in Γ_g or not.

(2). Determine the singular part of $\frac{1}{|\mathbb{M}|} \sum_{g \in \Phi_c} \chi(g) t_g(z)$ at c.

We will investigate those questions in the next section.

3. EQUIVALENCE OF CUSPS

In this section, we will study the equivalence of cusps in Γ_g, $g \in \mathbb{M}$.

Lemma 1. *For each exact divisor e of N and for each c such that $gcd(c, e) = 1$, $\Gamma_0(N)$ admits an Atkin-Lehner involution of the form $W_e = \begin{pmatrix} ae & b \\ cN & de \end{pmatrix}$. Moreover, one can choose either $a = 1$ or $d = 1$ if desired.*

Proof. For each c such that c and e are relatively prime, we have $\gcd(\frac{cN}{e}, e) = 1$. Hence, there exists b and y such that $ye - \frac{bcN}{e} = 1$, or $ye^2 - bcN = e$. The lemma follows by writing y into ad for suitable a and d. □

Lemma 2. *Let $gcd(a, b) = 1$ and M be nonzero integers. Then there exists a pair of integers x and y satisfying $gcd(xM, y) = 1$ and $ax + by = 1$.*

Proof. This is a well known fact of the elementary number theory. Let x' and y' be a pair of integral solutions of the equation $ax+by = 1$ and let $M = M_a M_{y'} M'$ be the decomposition of M into a product of coprime factors such that M_a and a, $M_{y'}$ and y', have the same prime factors. It is clear that $y = y' + aM'$ and $x = x' - bM'$ is also a pair of solutions to the equation. Note that $\gcd(x, y) = 1$ since it is a solution of $ax+by = 1$. Furthermore, one has $\gcd(y, M) = \gcd(y'+aM', M_a M_{y'} M')$ $= \gcd(y' + aM', M') = 1$. Therefore x and y is pair of integral solutions of the equation such that $\gcd(xM, y) = 1$. □

Lemma 3. *Let $\frac{x}{y}$, gcd(x,y) = 1, be a rational number. Then $\frac{x}{y}$ is equivalent to some $\frac{x'}{y'}$, gcd(x',y') = 1 in $\Gamma_0(N)$, where y' = gcd(N,y). Furthermore, if $\frac{x}{y}$ is equivalent to $\frac{x''}{y''}$ with $y''|N$, $y'' > 0$, and $gcd(x'', y'') = 1$, then $y'' = y'$.*

Proof. Consider the equality

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \frac{x}{y} = \frac{ax + by}{cNx + dy} = \frac{ax + dy}{y'(cx\frac{N}{y'} + d\frac{y}{y'})}$$

Note that $\gcd(x\frac{N}{y'}, \frac{y}{y'}) = 1$, hence the equation

$$c\frac{xN}{y'} + d\frac{y}{y'} = 1$$

is solvable for c, d in $\mathbb{Z}$. Applying Lemma 2, we may assume that c and d are integral solutions of the above equation such that $\gcd(cN, d) = 1$. Let a and b be chosen such that $ad - cbN = 1$. Summerizing, we now conclude that $\frac{x}{y}$ is equivalent to $\frac{ax+by}{y'}$ by $\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$. Since $\gcd(ax + by, y') = \gcd(ax, y') = \gcd(a, y') = 1$, first part of the lemma is proved. As for the second part, suppose

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \frac{x}{y} = \frac{ax + by}{cNx + dy} = \frac{ax + dy}{y'(cx\frac{N}{y'} + d\frac{y}{y'})} = \frac{x''}{y''}.$$

We first note $y'|y''$, since $\gcd(ax + by, y') = 1$. To show $y''|y'$, suppose that y'' possesses a prime power q^t such that y' is not a multiple of q^t, then $q^t|y'(cx\frac{N}{y'} + d\frac{y}{y'})$

implies $q|(cx\frac{N}{y'}+d\frac{y}{y'})$. Since $y''|N$, q is a divisor of $\frac{N}{y'}$, hence $q|d$. This implies that $\gcd(cN,d)\neq 1$, against our choice of c, d. Thus, $y''|y'$ and the second part of the lemma is proved. $\square$

In Lemma 4 and Lemma 5, $G = n|h+e,f,\cdots$ is the eigen group of the invariance group Γ_g.

Lemma 4. *Let $g\in\mathbb{M}$ and let $\Gamma_g \le G = n|h+e,f,\cdots$ be the invariance group of $t_g(z)$. Then $G\infty = \Gamma_g\infty$.*

Proof. Since $q=\exp(2\pi i z)$ is a local parameter of $t_g(z)$, the stabilizer $(\Gamma_g)_\infty$ of ∞ is generated by $\begin{pmatrix}1&1\\0&1\end{pmatrix}$. As for G, G_∞ is generated by $\begin{pmatrix}1&\frac{1}{h}\\0&1\end{pmatrix}$. Hence

$$[G:\Gamma_g]=h=[G_\infty:(\Gamma_g)_\infty]$$

Consequently, $G\infty=\Gamma_g\infty$. $\square$

Lemma 5. *Let $g\in\mathbb{M}$ and let $\Gamma_g\le n|h+e,f,\cdots$ be the invariance group of $t_g(z)$. Then $\frac{x}{y}$, $gcd(x,y)=1$, is equivalent to ∞ in Γ_g if and only if*

$$gcd(\frac{n}{h},\frac{y}{gcd(y,h)})\in\left\{\frac{n}{h},\frac{n}{he},\frac{n}{hf},\cdots\right\}$$

Proof. To simplify our notation, let $\mathrm{N}=\frac{n}{h}$. Choose the Atkin-Lehner involution W_e as described in Lemma 1 with $\gcd(c,N)=1$. One has $W_e\infty=\frac{a}{c\frac{N}{e}}$, and

$$\gcd(a,c\frac{N}{e})=1.$$

By Lemma 3, there exists $\gamma_e\in\Gamma_0(N)$ such that $\gamma_eW_e\infty=\frac{e'}{\frac{N}{e}}$ where $\gcd(e',\frac{N}{e})=1$, since $\gcd(c\frac{N}{e},N)=\frac{N}{e}$. Note that e is an exact divisor of N, hence among the representitives of inequivalent cusps of $\Gamma_0(N)$, there is exactly one and only one cusp z with denominator $\frac{N}{e}$ (see Harada [4]). Without loss of generality, we may assume that $z=\frac{1}{\frac{N}{e}}$. Therefore, we may assume that γ_e is chosen so that $\gamma_eW_e\infty=\frac{1}{\frac{N}{e}}$. Hence the G-orbit of ∞ can be decomposed into,

$$\begin{pmatrix}\frac{1}{h}&0\\0&1\end{pmatrix}\Gamma_0(N)\frac{1}{N}\cup\begin{pmatrix}\frac{1}{h}&0\\0&1\end{pmatrix}\Gamma_0(N)\frac{1}{\frac{N}{e}}\cup\begin{pmatrix}\frac{1}{h}&0\\0&1\end{pmatrix}\Gamma_0(N)\frac{1}{\frac{N}{f}}\cup\cdots.$$

Hence $\frac{x}{y}$ is equivalent to ∞ in G if and only if

$$\begin{pmatrix}h&0\\0&1\end{pmatrix}\frac{x}{y}=\frac{\frac{hx}{\gcd(y,h)}}{\frac{y}{\gcd(y,h)}}\in\Gamma_0(N)\frac{1}{N}\cup\Gamma_0(N)\frac{1}{\frac{N}{e}}\cup\Gamma_0(N)\frac{1}{\frac{N}{f}}\cup\cdots,$$

which is equivalent to, by Lemma 3,

$$\gcd(\frac{y}{\gcd(y,h)},N)\in\left\{N,\frac{N}{e},\frac{N}{f},\cdots\right\}.$$

$G\infty=\Gamma_g\infty$ as shown in Lemma 4 and so $\frac{x}{y}$ is equivalent to ∞ in Γ_g if and only if

$$\gcd(\frac{y}{\gcd(y,h)},\frac{n}{h})\in\left\{\frac{n}{h},\frac{n}{he},\frac{n}{hf},\cdots\right\}.$$

$\square$

Corollary 6. *$0 = \frac{0}{1}$ is equivalent to ∞ in Γ_g if and only if $n = h$ or $G = n|h + e, f, \cdots$ admits the Atkin-Lehner involution $W_{\frac{n}{h}}$.*

Proof. Since $\gcd(1, \frac{n}{h}) = 1$, G must admits an Atkin-Lehner involution W_e such that $\frac{n}{he} = 1$, hence $e = \frac{n}{h}$. □

Let χ be an irreducible character of the monster $\mathbb{M}$. In order to determine the singular part of

$$\frac{1}{|\mathbb{M}|} \sum_{g \in \Phi_c} \chi(g) t_g(z)$$

at the cusp $c = \frac{x}{y}$, where $\gcd(x, y) = 1$ and $y|N_\chi$, it is necessary to find a matrix P_c in $SL_2(\mathbb{R})$ such that $P_c\infty = c$ and determine the q-expansion of

$$\frac{1}{|\mathbb{M}|} \sum_{g \in \Phi_c} \chi(g)(t_g|P_c)(z),$$

which will be called the q-expansion of $t_\chi(z)$ at c. Such a matrix P_c is easy to find and choice is not unique. To ease the computation of the q-expansion of

$$\frac{1}{|\mathbb{M}|} \sum_{g \in \Phi_c} \chi(g)(t_g|P_c)(z),$$

it is necessary to find a good P_c so that the transformation formula of $t_g|P_c$ can be obtained for every g in Φ_c simultaneously. What we shall do is as follows. Namely, we will find a matrix P_c so that one can associate with P_c an upper triangular matrix $U_{c,g}$ such that

$$P_c {U_{c,g}}^{-1} \in n_g|h_g + e_g, f_g, \cdots.$$

Since $n_g|h_g + e_g, f_g, \cdots$ is the eigen group of Γ_g, elements in $n_g|h_g + e_g, f_g, \cdots$ map t_g to $\sigma_g t_g$ where σ_g is an h_g-th root of unity which depends on g and on some other quantities. Therefore

$$t_g|P_c = \sigma_g t_g|U_{c,g}.$$

A good transformation formula for $t_g|P_c$ is obtained since $U_{c,g}$ is upper trianglar.

Let y_0 be the exact divisor of N_χ such that y and y_0 share the same prime divisors. Then $\gcd(y, x\frac{N_\chi}{y_0}) = 1$ and so there is a matrix $P_c \in SL_2(\mathbb{Z})$ of the form

$$P_c = \begin{pmatrix} x & w \\ y & \frac{zN_\chi}{y_0} \end{pmatrix}.$$

Lemma 4 implies that $\frac{x}{y}$ is equivalent to ∞ in Γ_g if and only if $\frac{x}{y}$ is equivalent to ∞ in the eigen group $n_g|h_g + e_g, f_g, \cdots$ of g and so by Lemma 5, $\frac{x}{y}$ is equivalent to ∞ in Γ_g if and only if $\gcd(\frac{n_g}{h_g}, \frac{y}{\gcd(y,h_g)}) \in \left\{\frac{n}{h}, \frac{n}{he}, \frac{n}{hf}, \cdots\right\}$. More precisely, $\frac{x}{y}$ is equivalent to ∞ by an element in $n_g|h_g$ if

$$\gcd(\frac{n_g}{h_g}, \frac{y}{\gcd(y, h_g)}) = \frac{n_g}{h_g}$$

and is equivalent to ∞ by an Atkin-Lehner involution W_{e_g} of $n_g|h_g + e_g, f_g, \cdots$ if

$$\frac{n_g}{h_g e_g} = \gcd(\frac{n_g}{h_g}, \frac{y}{\gcd(y, h_g)}) \in \left\{\frac{n}{he}, \frac{n}{hf}, \cdots\right\}.$$

Lemma 7. *Suppose that $gcd(\frac{n_g}{h_g}, \frac{y}{gcd(y,h_g)}) = \frac{n_g}{h_g}$. Let u_g be chosen so that $\frac{yu_g}{h_g} + \frac{zN_\chi gcd(h_g,y)}{y_0 h_g}$ is an integer. Then*

$$P_c U_{c,g}^{-1} = P_c \begin{pmatrix} \frac{h_g}{gcd(h_g,y)} & \frac{u_g}{h_g} \\ 0 & \frac{gcd(h_g,y)}{h_g} \end{pmatrix} \in n_g|h_g$$

where

$$U_{c,g} = \begin{pmatrix} \frac{gcd(h_g,y)}{h_g} & -\frac{u_g}{h_g} \\ 0 & \frac{h_g}{gcd(h_g,y)} \end{pmatrix}.$$

Proof. To show the existence of u_g, simply solve the equation

$$\frac{y}{\gcd(h_g,y)} u_g + \frac{N_\chi}{y_0} z \equiv 0 \pmod{\frac{h_g}{\gcd(h_g,y)}}.$$

Then $yu_g + z\frac{N_\chi}{y_0}\gcd(h_g,y) \equiv 0 \pmod{h_g}$ and hence $\frac{yu_g}{h_g} + \frac{zN_\chi \gcd(h_g,y)}{y_0 h_g}$ is an integer.

The matrix

$$P_c \begin{pmatrix} \frac{h_g}{\gcd(h_g,y)} & \frac{u_g}{h_g} \\ 0 & \frac{\gcd(h_g,y)}{h_g} \end{pmatrix} = \begin{pmatrix} \frac{xh_g}{\gcd(h_g,y)} & \frac{xu_g + w\gcd(h_g,y)}{h_g} \\ \frac{yh_g}{\gcd(h_g,y)} & \frac{yu_g}{h_g} + \frac{zN_\chi \gcd(h_g,y)}{y_0 h_g} \end{pmatrix}$$

has the property

(1). $\frac{yu_g}{h_g} + \frac{zN_\chi \gcd(h_g,y)}{y_0 h_g}$ is an integer by our choice of u_g, and,

(2). $\frac{yh_g}{\gcd(h_g,y)}$ is a multiple of n_g since $\gcd(\frac{n_g}{h_g}, \frac{y}{\gcd(y,h_g)}) = \frac{n_g}{h_g}$.

Therefore

$$P_c U_{c,g}^{-1} = P_c \begin{pmatrix} \frac{h_g}{\gcd(h_g,y)} & \frac{u_g}{h_g} \\ 0 & \frac{\gcd(h_g,y)}{h_g} \end{pmatrix} \in n_g|h_g. \quad \square$$

Corollary 8. *Suppose that $gcd(\frac{n_g}{h_g}, \frac{y}{gcd(y,h_g)}) = \frac{n_g}{h_g}$. Then*

$$t_g|P_c = \sigma_g t_g|U_{c,g} = \sigma_g t_g(U_{c,g} z) = \sigma_g t_g(\frac{gcd(h_g,y)^2}{{h_g}^2} z - \frac{u_g}{h_g^2} gcd(h_g,y))$$

where σ_g is an h_g-th root of unity.

Proof. Since $P_c \begin{pmatrix} \frac{h_g}{\gcd(h_g,y)} & \frac{u_g}{h_g} \\ 0 & \frac{\gcd(h_g,y)}{h_g} \end{pmatrix} \in n_g|h_g$ and $n_g|h_g$ is the eigen group of $t_g(z)$

$$t_g|P_c \begin{pmatrix} \frac{h_g}{\gcd(h_g,y)} & \frac{u_g}{h_g} \\ 0 & \frac{\gcd(h_g,y)}{h_g} \end{pmatrix} = \sigma_g t_g(z).$$

This completes the proof of the corollary. $\square$

We now consider the case that c is equivalent to ∞ in the eigen group of Γ_g by an Atkin-Lehner involution W_{e_g}.

Lemma 9. *Suppose that $c = \frac{x}{y}$ is equivalent to ∞ in the eigen group of Γ_g by an Atkin-Lehner involution W_{e_g}. Let an integer u_g be chosen such that e_g is a divisor of an integer $\frac{u_g y}{h_g} + \frac{zN_\chi gcd(h_g,y)}{h_g y_0}$ where*

$$e_g = \frac{\frac{n_g}{h_g}}{gcd(\frac{n_g}{h_g}, \frac{y}{gcd(y,h_g)})}.$$

Then

$$P_c U_{c,g}^{-1} = P_c \begin{pmatrix} \frac{e_g h_g}{gcd(h_g,y)} & \frac{u_g}{h_g} \\ 0 & \frac{gcd(h_g,y)}{h_g} \end{pmatrix} = W_{e_g} \in n_g|h_g + e_g, f_g, \cdots.$$

Furthermore,

$$t_g|P_c = \sigma_g t_g(U_{c,g}z) = \sigma_g t_g(\frac{gcd(h_g,y)^2}{e_g {h_g}^2} z - \frac{u_g}{e_g h_g^2} gcd(h_g,y))$$

where σ_g is an h_g-th root of unity.

Proof. Let us first show that such an u_g exists. We will need u_g such that

$$yu_g + z\frac{N_\chi}{y_0}\mathrm{gcd}(h_g,y) \equiv 0 \pmod{e_g h_g}.$$

This follows from

$$\frac{y}{\mathrm{gcd}(h_g,y)}u_g + z\frac{N_\chi}{y_0} \equiv 0 \pmod{\frac{h_g}{\mathrm{gcd}(h_g,y)}e_g}.$$

Since $\mathrm{gcd}(\frac{n_g}{h_g}, \frac{y}{\mathrm{gcd}(h_g,y)}) = \frac{n_g}{h_g e_g}$ and e_g is an exact divisor of $\frac{n_g}{h_g}$, we see that

$$\mathrm{gcd}(\frac{y}{\mathrm{gcd}(h_g,y)}, e_g) = 1.$$

Therefore $\frac{y}{\mathrm{gcd}(h_g,y)}$ is invertible modulo $\frac{h_g}{\mathrm{gcd}(h_g,y)}e_g$, hence u_g exists as required.

The matrix

$$P_c \begin{pmatrix} \frac{e_g h_g}{\mathrm{gcd}(h_g,y)} & \frac{u_g}{h_g} \\ 0 & \frac{\mathrm{gcd}(h_g,y)}{h_g} \end{pmatrix} = \begin{pmatrix} \frac{xe_g h_g}{\mathrm{gcd}(h_g,y)} & \frac{xu_g + w\mathrm{gcd}(h_g,y)}{h_g} \\ \frac{ye_g h_g}{\mathrm{gcd}(h_g,y)} & \frac{yu_g}{h_g} + \frac{zN_\chi \mathrm{gcd}(h_g,y)}{y_0 h_g} \end{pmatrix}$$

has the property

(1). $\frac{u_g y}{h_g} + \frac{zN_\chi \mathrm{gcd}(h_g,y)}{h_g y_0}$ is a multiple of e_g by choice of u_g, and,

(2). $\frac{ye_g h_g}{\mathrm{gcd}(h_g,y)}$ is a multiple of n_g, since $\mathrm{gcd}(\frac{n_g}{h_g}, \frac{y}{\mathrm{gcd}(y,h_g)}) = \frac{n_g}{h_g e_g}$.

Therefore

$$P_c \begin{pmatrix} \frac{e_g h_g}{\mathrm{gcd}(h_g,y)} & \frac{u_g}{h_g} \\ 0 & \frac{\mathrm{gcd}(h_g,y)}{h_g} \end{pmatrix} = W_{e_g} \in n_g|h_g + e_g, f_g, \cdots.$$

Since c is equivalent to ∞ in the eigen group of Γ_g by W_{e_g}, the transformation formula follows easily. □

Remark. It is easy to see that Lemma 7 and Corollary 8 are included in Lemma 9 if $e_g = 1$, in which case every element of $n_g|h_g$ is called an Atkin-Lehner involution for $e_g = 1$. This abuse of words will be used occasionally for the balance of the paper.

The singular part of $\frac{1}{|\mathbb{M}|}\sum_{g\in\Phi_c}\chi(g)(t_g|P_c)(z)$ at $z=\infty i$ is now determined by

$$\mathrm{sing}_{P_c}t_\chi = \frac{1}{|\mathbb{M}|}\sum_{g\in\Phi_c}\chi(g)\frac{\sigma_g}{e^{2\pi i U_{c,g}z}}.$$

We give a few examples in the calculation of $\mathrm{sing}_{P_c}t_\chi$'s. Note that the first example will be used later in the determination of the invariance groups.

Example 1. Suppose $c=\frac{0}{1}$. Then $\mathrm{sing}_{P_0}t_\chi = \frac{1}{|\mathbb{M}|}\sum_{g\in\Phi_c}\chi(g)\frac{\sigma_g}{q^{\frac{1}{n_g h_g}}}$.

Proof. In this case $y=1$. We may choose $P_0 = \begin{pmatrix} x & w \\ y & \frac{xN_\chi}{y_0}\end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$ and $u_g=0$. If the condition of Corollary 8 holds, then $n_g=h_g$ and

$$t_g|P_0 = \sigma_g t_g(\frac{z}{h_g^2}) = \sigma_g t_g(\frac{z}{n_g h_g}).$$

The only $g\in\Phi_0\subseteq\mathbb{M}$ satisfying $n_g=h_g$ are $1A$ and $3C$. We have

$$t_{1A}|P_0 = t_{1A} \quad\text{and}\quad t_{3C}|P_0=\sigma_{3C}t_{3C}(\frac{1}{9}z).$$

On the other hand, if the condition of Lemma 9 holds, then $n_g=e_g h_g$ and

$$t_g|P_0 = \sigma_g t_g(\frac{z}{e_g h_g^2}) = \sigma_g t_g(\frac{z}{n_g h_g}).$$

Note that for all the remaining $g\in\Phi_0\backslash\{1A,3C\}\subseteq\mathbb{M}$, 0 is equivalent to ∞ in Γ_g by the Atkin-Lehner involution $W_{\frac{n_g}{h_g}}$ and hence the condition of Lemma 9 holds. □

Remark. Example 1 shows that 0 is a pole of the McKay-Thompson series $t_\chi(z)$ for every χ, since $n_g h_g\neq 1$ for $g\neq 1$ and so the coefficient of q^{-1} is nonzero.

Example 2. Let χ be the trivial character and let $c=\frac{1}{3}$. Then $P_{\frac{1}{3}} = \begin{pmatrix} 1 & [\frac{N_0}{81}] \\ 3 & \frac{N_0}{27}\end{pmatrix}$ ($[x]$ is the integral part of x and $\frac{N_0}{27}\equiv 1 \pmod 3$) and $t_{84A}|P_{\frac{1}{3}}$ $=\sigma_{84A}t_{84A}(\frac{1}{56}z+\frac{1}{2})$.

Proof. We know $\Gamma_{84A} < 84|2+$. Since $\gcd(\frac{84}{2},\frac{3}{\gcd(3,2)}) = 3 = \frac{84}{2e}$, we see that e is 14, and can choose $u_g=-28$.

$$P_{\frac{1}{3}}\begin{pmatrix} 28 & -\frac{28}{2} \\ 0 & \frac{1}{2}\end{pmatrix} = W_{14}\in 84|2+.$$

The rest follows easily. □

Remark. (1). The invariance group Γ_g of the harmonics $n|h+e,f,\cdots$ are not fully determined. (For each g, one can write down a set of generators of the invariance group Γ_g easily. But determining whether or not a given element in $SL_2(\mathbb{R})$ is a word of those generators is nontrivial.) Hence we have to settle for σ_g being an h_g-th root of unity.

(2). $\sigma_g=1$ if $h_g=1$.

(3). Let $p\in\{11,17,19,23,29,31,41,47,59,71\}$. Applying Lemma 5, one can prove $\Phi_0=\Phi_{\frac{x}{p}}$ and $\Phi_{\frac{1}{32}}=\Phi_{\frac{1}{64}}$.

4. INVARIANCE GROUP

Let f be a modular function and let K_f be a subgroup of the invariance group (in $SL_2(\mathbb{R})$) Γ_f of f of finite index. We shall determine the invariance group as follows. Define

C_f = the set of all cusps of K_f, and,
$C_0 = \{c \in C_f | c \text{ is a pole of } f\}$.

Lemma 10. *We have $\Gamma_f C_f \subseteq C_f$ and $\Gamma_f C_0 \subseteq C_0$.*

Proof. Since $[\Gamma_f : K_f] < \infty$, C_f is also the set of all cusps of Γ_f. The second statement is obvious. □

Lemma 11. *Let f be a modular function and let Γ_f be its invariance group. Suppose that $K_f \leq \Gamma_f$. Let $\alpha = \frac{a_1}{c_1}$ $(a_1 \neq 0)$ and $\beta = \frac{a_2}{c_2}$ be two inequivalent cusps of K_f. Let*

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \textit{and} \quad M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Then $\frac{a_1}{c_1}$ and $\frac{a_2}{c_2}$ are equivalent with respect to Γ_f if and only if the q-expansion of $f|_{M_1}$ is derived from that of $f|_{M_2}$ under the substitution $z \to az + b$ for some numbers a and b (if $c_i = 0$, then $a_i = 1$ and $\frac{a_i}{c_i} = \infty$).

Proof. Let $A \in \Gamma_f$ be such that $A\alpha = \beta$. Define the matrix B such that

$$A = M_2 \begin{pmatrix} 1 & 0 \\ -\frac{c_1}{a_1} & 1 \end{pmatrix} B.$$

Since $A\alpha = \beta$, it follows that $B\alpha = \alpha$. Hence

$$B = \begin{pmatrix} 1 & 0 \\ \frac{c_1}{a_1} & 1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c_1}{a_1} & 1 \end{pmatrix}$$

for some m_{11}, m_{12} and m_{22}. In particular,

$$A = M_2 \begin{pmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c_1}{a_1} & 1 \end{pmatrix}$$

and

$$A\alpha = M_2 \begin{pmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{pmatrix} \infty.$$

It follows that $f|_{M_1} = f|_{AM_1} =$

$$f|M_2 \begin{pmatrix} 1 & 0 \\ -\frac{c_1}{a_1} & 1 \end{pmatrix} BM_1 = f|M_2 \begin{pmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & a' \end{pmatrix} = f|M_2 \begin{pmatrix} m'_{11} & m'_{12} \\ 0 & m'_{22} \end{pmatrix},$$

for some b_1, a', m'_{11}, m'_{12} and m'_{22}. Consequently, $f|_{M_1}$ is derived from that of $f|_{M_2}$ under the substitution $z \to az + b$ where $a = \frac{m'_{11}}{m'_{22}}$ and $b = \frac{m'_{12}}{m'_{22}}$. Conversely, one sees easily that α and β is equivalent to each other by

$$M_1 \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} M_2^{-1} \in \Gamma_f.$$

□

The invariance group Γ_f can now be determined as follows :

(1). Determine C_f, the set of all cusps of K_f.
(2). Determine the subset C_0.
(3). Determine the q-expansion of f at all c_i in C_0 by suitable matrices M_i such that $M_i\infty = c_i$.
(4). Apply Lemma 11 and determine the set $E_0 = \{c \in C_0|$ the q-expansion of f at c is derived from the q-expansion of f at 0 under the substitution $z \to az + b\}$ and the set $A_0 = \{A_c \in \Gamma_f, A_c0 = c\}$. Note that it is sufficient to determine at most one matrix A_c for each representative of inequivalent cusps.
(5). Determine $(\Gamma_f)_0 = \langle B|B = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}, m \in r\mathbb{Z},$ for some (fixed) $\searrow \in \mathbb{Q}\rangle$. Note that this can be achieved by investigating the q-expansion of f at 0. Note also that B is of the form given since Γ_f is discrete.

Remark. (1). The McKay-Thompsom series $t_\chi(z)$ has a pole at 0 for every χ as stated in the remark right after Example 1.
(2). Since Lemma 11 applies only when one of the cusp is nonzero, one can not take c to be 0 in (4) above.
(3). One can replace 0 by any cusp and apply our procedure to find the invariance groups.

Lemma 12. $\Gamma_f = \langle K_f, B, A_c, c \in E_0\rangle$.

Proof. For any $\sigma \in \Gamma_f \backslash \langle K_f, A_c, c \in E_0\rangle$. Applying Lemma 10, $\sigma 0$ is again a cusp. Hence $\sigma 0$ must be $\langle K_f, A_c, c \in E_0\rangle$-equivalent to 0. Choose $\delta \in \langle K_f, A_c, c \in E_0\rangle$ such that $\delta\sigma 0 = 0$. Then $\delta\sigma \in (\Gamma_f)_0$. Hence $\Gamma_f = \langle K_f, B, A_c, c \in E_0\rangle$ holds. □

Theorem 13. (Helling's Theorem [5]) *The maximal discrete groups of $PSL(2,\mathbb{C})$ commensurable with the modular group $SL(2,\mathbb{Z})$ are just the images of the conjugates of $\Gamma_0(N)+$ for square free N.*

Corollary 14. *For each irreducible character χ of $\mathbb{M}$, the set of prime divisors of the index $[\Gamma_\chi : \Gamma_0(N_\chi)]$ is a subset of $\{2, 3, 5, 7\}$.*

Proof. By Helling's Theorem, any maximal subgroup that contains Γ_χ is a conjugate of some $\Gamma_0(n)+$. Conway has shown in [1] that n must be a divisor of N_χ. Now compare the volumes of the fundamental domains of $SL_2(\mathbb{Z})$, $\Gamma_0(n)$, and $\Gamma_0(N_\chi)$. Noting that the conjugation does not change the volume and that the normalizer of $\Gamma_0(n)$ changes the volume of the fundamental domain by a factor involving only primes 2 and 3, we obtain our lemma since the index $[SL_2(\mathbb{Z}) : \Gamma_{\not{\vdash}}(\mathbb{N}_\chi)]$ involves only primes 2, 3, 5, and 7. □

Let χ be an irreducible character of $\mathbb{M}$ and Γ_χ be the invariance group of $t_\chi(z)$. We are now ready to prove :

(1). $A_0 = \emptyset$, and,
(2). $(\Gamma_\chi)_0 = (\Gamma_0(N_\chi))_0$.

Lemma 15. *Let χ be an irreducible character of $\mathbb{M}$ and let c be a cusp of $\Gamma_0(N_\chi)$, not equivalent to 0. Then $A_0 = \{A_c|A_c0 = c\} = \emptyset$.*

Proof. We first recall that the singular part of $\frac{1}{|\mathbb{M}|}\sum_{g\in\Phi_c}\chi(g)(t_g|P_c)(z)$ at $z=\infty i$ is given by

$$\mathrm{sing}_{P_c}t_\chi=\frac{1}{|\mathbb{M}|}\sum_{g\in\Phi_c}\chi(g)\frac{\sigma_g}{e^{2\pi iU_{c,g}z}}.$$

Applying Lemmas 7, 9 and 10 and Corollary 8, we see that it suffices to show that $\mathrm{sing}_{P_0}t_\chi$ can not be derived from $\mathrm{sing}_{P_c}t_\chi$ under the substitution $z\rightarrow az+b$ if $c\neq 0$. This is achieved by a *case-by-case* study. We give an example to indicate how the lemma is proved.

Example 3. Let χ be the trivial character of $\mathbb{M}$. Then $A_0=\emptyset$.

Proof. We first note that for any irreducible character χ of $\mathbb{M}$ and $c\in\mathbb{Q}\cup\{\infty\}$, we have :

(1). The lowest terms in $\mathrm{sing}_{P_c}t_\chi$ and $\mathrm{sing}_{P_0}t_\chi$ are of the form $\frac{r}{q}$ for some number $r\in\mathbb{Q}$, and

(2). Terms in $\mathrm{sing}_{P_c}t_\chi$ and $\mathrm{sing}_{P_0}t_\chi$ are all of the form $\frac{r}{q^{\frac{1}{t}}}$ (Corollary 8, Lemma 9), for some $r\in\mathbb{Q}$, and $t\in\mathbb{N}$.

Since the lowest term in $\mathrm{sing}_{P_0}t_\chi$ is $\frac{r}{q}$, $r\neq 0$, the transformation that sends $\mathrm{sing}_{P_c}t_\chi$ to $\mathrm{sing}_{P_0}t_\chi$ is of the form $z\rightarrow az+b$ where a is some positive integer.

Let χ is the trivial character, then by Corollary 8 and Lemma 9, one has

$$\mathrm{sing}_{P_0}t_\chi=\frac{1}{|\mathbb{M}|}\sum_{g\in\Phi_0}\frac{\sigma_g}{q^{\frac{1}{n_gh_g}}}=\frac{1}{|\mathbb{M}|}\Big(\frac{1}{q}+\frac{2|\mathbb{M}|}{|\mathbb{C}_{\mathbb{M}}(\ntriangleright\nVdash\mathbb{A})|}\frac{\sigma_{71A}}{q^{\frac{1}{71}}}+\cdots\Big),$$

and for any cusp $c=\frac{x}{y}$, $\gcd(x,y)=1$,

$$\mathrm{sing}_{P_c}t_\chi=\frac{1}{|\mathbb{M}|}\sum_{g\in\Phi_c}\frac{\sigma_g}{e^{2\pi iU_{c,g}z}}.$$

Suppose $\mathrm{sing}_{P_0}t_\chi$ can be derived from $\mathrm{sing}_{P_c}t_\chi$ under $z\rightarrow az+b$.

(3). We will show $\gcd(y,71)=1$. Suppose false. Then $71|y$ and $71A$, $71B\in\Phi_c$. Since $\frac{1}{q^{\frac{1}{71}}}$ appears in $\mathrm{sing}_{P_0}t_\chi$, there exists, by Lemma 11, some g in Φ_c such that

$$\frac{\sigma_g}{e^{2\pi iU_{c,g}z}}|(z\rightarrow az+b)=\frac{r}{q^{\frac{1}{71}}}\qquad(*)$$

where r is some constant. Hence

$$\frac{\gcd(h_g,y)^2a}{e_gh_g^2}=\frac{1}{71}$$

where

$$e_g=\frac{\frac{n_g}{h_g}}{\gcd(\frac{n_g}{h_g},\frac{y}{\gcd(y,h_g)})}.$$

Since a is an integer, this implies that $g=71A$ or $71B$. Since $\Gamma_{71A}=\Gamma_{71B}=71+$, we have $n_{71A}=n_{71B}=71$, $h_{71A}=h_{71B}=1$, $e_{71A}=e_{71B}=71$, and $a=1$. On the other hand, Corollary 8 implies $t_g|P_c=\sigma_gt_g(z-u_g)$ for $g=71A$ or $71B$. Hence the transformation $(*)$ can not be done. This forces

$$\gcd(y,71)=1.$$

(4). Since $\frac{1}{q^{\frac{1}{t}}}$, $t \in \{29, 41, 59, 92, 93, 94, 95, 104, 110, 119\}$ all appear with nonzero coefficients in $\text{sing}_{P_0} t_\chi$, we may similarly conclude that $\gcd(y, p) = 1$ for the other prime divisors of N_0.

Hence $\gcd(y, N_0) = 1$ and c is equivalent to 0. Consequently, $A_0 = \emptyset$.

Lemma 16. $(\Gamma_\chi)_0 = (\Gamma_0(N_\chi)_0$.

Proof. Suppose not. Applying Corollary 14, we see that $(\Gamma_\chi)_0$ contains $B_r = \begin{pmatrix} 1 & 0 \\ \frac{N_\chi}{r} & 1 \end{pmatrix}$ for $r = 2$, 3, 5 or 7. This implies that the cusp ∞ is equivalent to $\frac{r}{N_\chi}$ in Γ_χ. Therefore $\text{sing}_\infty t_\chi$ must be derived from $\text{sing}_{P_{\frac{r}{N_\chi}}} t_\chi$ under the substitution $z \to az + b$. We can now apply an analogous procedure (using $y = \frac{r}{N_\chi}$) as in Example 3 to get a contradiction. $\square$

Remark. One can also prove Lemma 16 by claiming that B_r does not leave $t_\chi(z)$ invariant. Note that it is easy to show the claim since B_r leaves most of the $t_g(z)$ invariant except for those g's such that $n_g h_g$ is not a divisor of $\frac{N_\chi}{r}$.

Combining Lemma 15 and 16, we have :

Theorem 17. *Let χ be an irreducible character of $\mathbb{M}$. Then $\Gamma_\chi = \Gamma_0(N_\chi)$.*

N_χ can be found in *Table* 1.

Remark. (1). In Lemma 15 and 16, 0 is a better choice than the other cusps (∞, for example) since among all the $\text{sing}_{P_c} t_\chi$'s, $\text{sing}_{P_0} t_\chi$ is the one that involves most nonzero terms.

(2). N_χ and its prime decomposition is calculated by a software called GAP.

Table 1

χ_i	N_{χ_i}	N_{χ_i} (*prime decomposition*)
1	2331309585756753201600	$2^6 3^3 5^2 7.11.13.17.19.23.29.31.41.47.59.71$
2	11841091337275200	$2^6 3^3 5^2 7.11.13.17.19.23.29.31.41$
3	437868837806400	$2^6 3^3 5^2 7.11.13.17.19.23.29.47$
4	467584848090400	$2^5 3^3 5^2 7.11.13.17.19.23.41.71$
5	38732026132800	$2^6 3^3 5^2 7.11.13.17.19.47.59$
6	20350725595200	$2^6 3^3 5^2 7.11.13.17.19.31.47$
7	87358471200	$2^5 3^2 5^2 7.11.13.17.23.31$
8	7820482269600	$2^5 3^2 5^2 7.11.13.17.29.31.71$
9	18526958049600	$2^6 3^2 5^2 7.11.13.23.29.41.47$
10	222987885120	$2^6 3^3 5.7.11.13.19.23.59$
11	8490081600	$2^6 3^2 5^2 7.11.13.19.31$
12	19445025600	$2^6 3^2 5^2 7.11.13.19.71$
13	9958865716800	$2^6 3^3 5^2 7.11.13.17.19.23.31$
14	73513400	$2^5 3^3 5.7.11.13.17$
15	2244077793757800	$2^3 3^3 5^2 7.11.13.17.19.23.29.41.47$
16	3749442460305984	$2^6 3^3 13.23.29.31.41.47.59.71$
17	3749442460305984	$2^6 3^3 13.23.29.31.41.47.59.71$
18	726818400	$2^5 3^3 5^2 7.11.19.23$
19	9182927033280	$2^6 3^3 5.7.11.13.19.29.41.47$
20	35703027360	$2^5 3^2 5.7.11.13.17.31.47$
21	98066928960	$2^6 3^3 5.7.11.13.17.23.29$
22	22789166400	$2^6 3^3 5^2 7.11.13.17.31$
23	451392480	$2^5 3^3 5.7.11.23.59$
24	295495200	$2^5 3^2 5^2 7.11.13.41$
25	253955520	$2^6 3^3 5.7.13.17.19$
26	479256378753600	$2^6 3^2 5^2 7.19.31.41.47.59.71$
27	479256378753600	$2^6 3^2 5^2 7.19.31.41.47.59.71$
28	27003936960	$2^6 3^2 5.7.11.13.17.19.29$
29	81995760	$2^4 3^3 5.7.11.17.29$
30	69618669120	$2^6 3^2 5.7.11.13.19.31.41$
31	21416915520	$2^6 3^2 5.7.11.13.17.19.23$
32	214885440	$2^6 3^3 5.7.11.17.19$
33	2882880	$2^6 3^2 5.7.11.13$
34	332640	$2^5 3^3 5.7.11$
35	786240	$2^6 3^3 5.7.13$
36	11147099040	$2^5 3^3 5.7.11.23.31.47$
37	331962190560	$2^5 3^2 5.7.11.13.17.19.23.31$
38	333637920	$2^5 3^3 5.7.11.17.59$
39	845013600	$2^5 3^3 5^2 19.29.71$
40	845013600	$2^5 3^3 5^2 19.29.71$
41	16676856385200	$2^4 3^3 5^2 11.23.31.47.59.71$
42	16676856385200	$2^4 3^3 5^2 11.23.31.47.59.71$
43	186902100	$2^2 3^3 5^2 7.11.29.31$
44	46955594400	$2^5 3.5^2 11.13.41.47.71$

45	46955594400	$2^5 3.5^2 11.13.41.47.71$
46	54880846020	$2^2 3^2 5.7.11.13.17.19.23.41$
47	105386400	$2^5 3^3 5^2 7.17.41$
48	105386400	$2^5 3^3 5^2 7.17.41$
49	49584815280	$2^4 3^3 5.7.11.13.17.19.71$
50	6404580	$2^2 3^2 5.7.13.17.23$
51	12916800	$2^6 3^3 5^2 13.23$
52	646027200	$2^6 3^2 5^2 7.13.17.29$
53	228731328	$2^6 3^2 7.17.47.71$
54	228731328	$2^6 3^2 7.17.47.71$
55	19044013248	$2^6 3^3 13.23.29.31.41$
56	10944013248	$2^6 3^3 13.23.29.31.41$
57	25077360	$2^4 3.5.7.11.23.59$
58	198918720	$2^6 3^3 5.7.11.13.23$
59	19433872080	$2^4 3^3 5.7.23.29.41.47$
60	19433872080	$2^4 3^3 5.7.23.29.41.47$
61	2784600	$2^3 3^2 5^2 7.13.17$
62	245044800	$2^6 3^2 5^2 7.11.13.17$
63	57266969760	$2^5 3^3 5.7.11.13.17.19.41$
64	157477320	$2^3 3^2 5.7.11.13.19.23$
65	818809200	$2^4 3^2 5^2 11.23.29.31$
66	263877213600	$2^5 3^2 5^2 7.11.13.19.41.47$
67	1588466880	$2^6 3^2 5.7.11.13.19.29$
68	33005280	$2^5 3.5.7.11.19.47$
69	937440	$2^5 3^3 5.7.31$
70	32864832	$2^6 3^3 7.11.13.19$
71	182584514112	$2^6 3.7.11.13.17.29.41.47$
72	182584514112	$2^6 3.7.11.13.17.29.41.47$
73	982080	$2^6 3^2 5.11.31$
74	33542208	$2^6 3^3 7.47.59$
75	33542208	$2^6 3^3 7.47.59$
76	7650720	$2^5 3^3 5.7.11.23$
77	931170240	$2^6 3^2 5.7.11.13.17.19$
78	33754921200	$2^4 3^2 5^2 7.11.13.17.19.29$
79	42325920	$2^5 3^2 5.7.13.17.19$
80	4969440	$2^5 3^2 5.7.17.29$
81	63126554400	$2^5 3^2 5^2 7.13.23.59.71$
82	63126554400	$2^5 3^2 5^2 7.13.23.59.71$
83	208304928	$2^5 3^2 13.23.41.59$
84	208304928	$2^5 3^2 13.23.41.59$
85	704223936	$2^6 3^3 13.23.29.47$
86	704223936	$2^6 3^3 13.23.29.47$
87	1235520	$2^6 3^3 5.11.13$
88	3967200	$2^5 3^2 5^2 19.29$
89	11737440	$2^5 3^3 5.11.13.19$
90	11737440	$2^5 3^3 5.11.13.19$
91	2542811040	$2^5 3^2 5.7.11.17.19.71$
92	22102080	$2^6 3.5.7.11.13.23$
93	1441440	$2^5 3^2 5.7.11.13$
94	879840	$2^5 3^2 5.13.47$

95	16576560	$2^4 3^2 5.7.11.13.23$
96	21677040	$2^4 3^2 5.7.11.17.23$
97	5267201940	$2^2 3^2 5.7.11.13.23.31.41$
98	7900200	$2^3 3^3 5^2 7.11.19$
99	1660401600	$2^6 3.5^2 11.13.41.59$
100	1660401600	$2^6 3.5^2 11.13.41.59$
101	932769600	$2^6 3.5^2 7.17.23.71$
102	7601451872175	$3^3 5^2 7.17.19.29.41.59.71$
103	7601451872175	$3^3 5^2 7.17.19.29.41.59.71$
104	6511680	$2^6 3^2 5.7.17.19$
105	5844589984800	$2^5 3^2 5^2 7.19.31.47.59.71$
106	5844589984800	$2^5 3^2 5^2 7.19.31.47.59.71$
107	2434219200	$2^6 3^2 5^2 7.19.31.41$
108	2434219200	$2^6 3^2 5^2 7.19.31.41$
109	280800	$2^5 3^3 5^2 13$
110	947520	$2^6 3^2 5.7.47$
111	1016747424	$2^5 3^3 7.11.17.29.31$
112	386100	$2^2 3^3 5^2 11.13$
113	6568800	$2^5 3.5^2 7.17.23$
114	1374912	$2^6 3^2 7.11.31$
115	151200	$2^5 3^3 5^2 7$
116	92400	$2^4 3.5^2 7.11$
117	411840	$2^6 3^2 5.11.13$
118	19562400	$2^5 3^2 5^2 11.13.19$
119	12524852340	$2^2 3^3 5.7.11.13.17.29.47$
120	41801760	$2^5 3^2 5.7.11.13.29$
121	75698280	$2^3 3^3 5.7.17.19.31$
122	8148853440	$2^6 3^2 5.7.13.17.31.59$
123	864175548600	$2^3 3^3 5^2 7.13.17.31.47.71$
124	3456702194400	$2^5 3^3 5^2 7.13.17.31.47.71$
125	3456702194400	$2^5 3^3 5^2 7.13.17.31.47.71$
126	119700	$2^2 3^2 5^2 7.19$
127	13695552	$2^6 3^2 13.31.59$
128	752016096	$2^5 3^3 13.23.41.71$
129	752016096	$2^5 3^3 13.23.41.71$
130	19320840	$2^3 3^2 5.7.11.17.41$
131	1004683680	$2^5 3^2 5.7.11.13.17.41$
132	164160	$2^6 3^3 5.19$
133	14379596431200	$2^5 3^3 5^2 7.11.13.17.19.29.71$
134	14208480	$2^5 3^3 5.11.13.23$
135	497653200	$2^4 3^3 5^2 11.59.71$
136	497653200	$2^4 3^3 5^2 11.59.71$
137	995276700	$2^2 3^3 5^2 11.23.31.47$
138	5109369408	$2^6 3^3 11.13.23.29.31$
139	514080	$2^5 3^3 5.7.17$
140	59017104226080	$2^5 3^3 5.7.11.13.17.19.29.31.47$
141	1151710560	$2^5 3^2 5.7.11.13.17.47$
142	3767400	$2^3 3^2 5^2 7.13.23$
143	7600320	$2^6 3^2 5.7.13.29$
144	11823840	$2^5 3^3 5.7.17.23$

145	8558550	$2.3^2 5^2 7.11.13.19$
146	664020	$2^2 3^2 5.7.17.31$
147	4320	$2^5 3^3 5$
148	655188534	$2.3^3 7.11.13.17.23.31$
149	102240	$2^5 3^2 5.71$
150	157248	$2^6 3^3 7.13$
151	26489342880	$2^5 3^2 5.7.11.13.17.23.47$
152	93276960	$2^5 3.5.7.17.23.71$
153	13137600	$2^6 3.5^2 7.17.23$
154	428400	$2^4 3^2 5^2 7.17$
155	18221280	$2^5 3.5.7.11.17.29$
156	190072512	$2^6 3^2 7.17.47.59$
157	21176100	$2^2 3^3 5^2 11.23.31$
158	37130940	$2^2 3^3 5.7.11.19.47$
159	852390	$2.3^3 5.7.11.41$
160	184363200	$2^6 3^2 5^2 7.31.59$
161	108803771818560	$2^6 3^2 5.7.13.17.19.23.29.41.47$
162	1657656	$2^3 3^2 7.11.13.23$
163	345290400	$2^5 3^2 5^2 7.13.17.31$
164	90014400	$2^6 3^2 5^2 7.19.47$
165	30240	$2^5 3^3 5.7$
166	4032	$2^6 3^2 7$
167	4062240	$2^5 3^2 5.7.13.31$
168	3204801600	$2^6 3.5^2 7.11.13.23.29$
169	24196995900	$2^2 3^2 5^2 7.11.17.19.23.47$
170	668304	$2^4 3^3 7.13.17$
171	73920	$2^6 3.5.7.11$
172	6983776800	$2^5 3^3 5^2 7.11.13.17.19$
173	32959080	$2^3 3^2 5.7.11.29.41$
174	3115200	$2^6 3.5^2 11.59$
175	48163383908640	$2^5 3^2 5.7.11.13.17.19.31.47.71$
176	427211200	$2^6 5^2 13.19.23.47$
177	858816	$2^6 3^3 7.71$
178	21416915520	$2^6 3^2 5.7.11.13.17.19.23$
179	14400	$2^6 3^2 5^2$
180	14400	$2^6 3^2 5^2$
181	154881891350	$2.3^3 5^2 11.13.19.29.31.47$
182	2009280	$2^6 3.5.7.13.23$
183	6339168	$2^5 3^3 11.23.29$
184	26429760	$2^6 3^3 5.7.19.23$
185	32730048	$2^6 3^3 13.31.47$
186	7425600	$2^6 3.5^2 7.13.17$
187	8237275200	$2^6 3^2 5^2 7.11.17.19.23$
188	15120	$2^4 3^3 5.7$
189	54774720	$2^6 3^2 5.7.11.13.19$
190	27989280	$2^5 3^3 5.11.19.31$
191	34272	$2^5 3^2 7.17$
192	3500640	$2^5 3^2 5.11.13.17$
193	1049200425	$3^3 5^2 7.13.19.29.31$
194	1404480	$2^6 3.5.7.11.19$

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